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# The Bethe ansatz in a periodic box-ball system and the ultradiscrete Riemann theta function

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ABSTRACT. Vertex models with quantum group symmetry give rise to integrable cellular automata at $q = 0$. We study a prototype example known as the periodic box-ball system. The initial value problem is solved in terms of an ultradiscrete analogue of the Riemann theta function whose period matrix originates in the Bethe ansatz at $q = 0$.

## 1. Introduction

The periodic box-ball system [10, 12] is a completely integrable one-dimensional cellular automaton. Its dynamics is described as a motion of balls hopping exclusively along the periodical array of boxes having capacity 1. The system is identified with a solvable vertex model [2] associated with quantum affine algebra $U_q(\widehat{sl}_2)$ at $q = 0$, where the fusion transfer matrices $T_1, T_2, \ldots$ yield a commuting family of deterministic time evolutions.

In [10], the initial value problem of the periodic box-ball system is solved by an inverse scattering method. It is done by synthesizing the combinatorial versions of the Bethe ansatz [3] at $q = 1$ [8] and $q = 0$ [9]. The action-angle variables are introduced by generalizing the rigged configurations ($q = 1$) up to some equivalence specified by the string center equation ($q = 0$). It enables one to determine the time evolution $T_l^t(p)$ of any state $p$ by an explicit algorithm whose computational steps are independent of the time $t$.

The Bethe ansatz approach [10] captures several characteristic features in the quasi-periodic solutions of soliton equations [4, 5]. For instance, the original nonlinear dynamics becomes a straight motion of the Bethe roots (angle variable) which live in an ultradiscrete analogue (2.6) of the Jacobi variety.

In this paper we exploit such an analogy further by representing the solution of the initial value problem explicitly in terms of the ultradiscretization (UD) of the Riemann theta function ($\mathbf{z} \in \mathbb{R}^g$):

$$\begin{aligned}\Theta(\mathbf{z}) &= \lim_{\epsilon\to+0} \epsilon \log\left(\sum_{\mathbf{n}\in\mathbb{Z}^g} \exp\Big(-\frac{{}^t\mathbf{n}A\mathbf{n}/2 + {}^t\mathbf{n}\mathbf{z}}{\epsilon}\Big)\right) \\ &= -\min_{\mathbf{n}\in\mathbb{Z}^g}\{{}^t\mathbf{n}A\mathbf{n}/2 + {}^t\mathbf{n}\mathbf{z}\}.\end{aligned} \tag{1.1}$$

Here $A$ is the symmetric positive definite $g \times g$ integer matrix (2.5) appearing in the string center equation (4.1) introduced in [9]. Likewise the Riemann theta function, $\Theta(\mathbf{z})$ enjoys the quasi-periodicity:

$$\Theta(\mathbf{z}+\mathbf{v}) = {}^t\mathbf{v}A^{-1}(\mathbf{z}+\mathbf{v}/2) + \Theta(\mathbf{z}) \quad \text{for any } \mathbf{v} \in \Gamma = A\mathbb{Z}^g. \tag{1.2}$$

Let $c_L(\mathbf{n}) = {}^t\mathbf{n}A\mathbf{n}/2 + {}^t\mathbf{n}\mathbf{z}$ be the quadratic form appearing in (1.1), where $L$ denotes the system size that enters $A$ and $\mathbf{z}$ in our main formula (3.8). The ultradiscrete Riemann theta function $\Theta(\mathbf{z})$ can be spotted in the following degeneration

scheme:

$$
\begin{array}{ccc}
 & \displaystyle\sum_{\mathbf{n}\in\mathbb{Z}^g}\exp\big(-c_L(\mathbf{n})/\epsilon\big) & \\
L\to\infty \swarrow & & \searrow \mathrm{UD}\\
\displaystyle\sum_{\mathbf{n}\in\{0,1\}^g}\exp\big(-c(\mathbf{n})/\epsilon\big) & & -\min_{\mathbf{n}\in\mathbb{Z}^g}\{c_L(\mathbf{n})\}=\Theta(\mathbf{z})\\
\mathrm{UD}\searrow & & \swarrow L\to\infty\\
 & -\displaystyle\min_{\mathbf{n}\in\{0,1\}^g}\{c(\mathbf{n})\} &
\end{array}\tag{1.3}
$$

At the top there is the Riemann theta function, which degenerates into various objects. The UD procedure (1.1) for getting $\Theta(\mathbf{z})$ is the SE arrow from the top. Then in the limit $L\to\infty$, the minimum over $\mathbf{n}\in\mathbb{Z}^g$ shrinks down to that over $\mathbf{n}\in\{0,1\}^g$, which reduces $c_L(\mathbf{n})$ to its $L$-independent part $c(\mathbf{n})$. Consequently, $\Theta(\mathbf{z})$ tends to the bottom one in (1.3), which we call the *ultradiscrete tau function*. The resulting expression (3.2) for the infinite system gives the piecewise linear formula for the Kerov-Kirillov-Reshetikhin (KKR) bijection [8] from rigged configurations to highest paths. One may go down the diagram (1.3) via the other route. The thereby encountered function in the middle left is the sum of $2^g$ "trigonometric terms" that are characteristic in the tau functions of soliton solutions for the infinite system [7]. In fact a procedure analogous to the SW arrow from the top has been described in p3.253 in [11], where quasi-periodic soliton solutions tend to those in the infinite system.

In our approach, the ultradiscrete Riemann theta function $\Theta(\mathbf{z})$ arises most naturally by going from the bottom in (1.3) into the NE direction. The essential idea [10] is to embed a state $p$ of the periodic box-ball system into an infinite system as $p\otimes p\otimes p\otimes\cdots$. It turns out that the ultradiscrete tau function for such periodic states is nothing but $\Theta(\mathbf{z})$ up to irrelevant contributions. As an application we extend the problem to $(\mathbb{C}^2)^{\otimes L}$ and construct joint eigenvectors of the commuting time evolutions. The result may be viewed as an explicit formula of the Bethe vectors at $q=0$ in terms of the ultradiscrete Riemann theta function.

In section 2, we recall the periodic box-ball system and the inverse scattering algorithm that solves the initial value problem [10]. Section 3 contains our main theorem 3.3. Section 4 gives the discussion on the connection with the Bethe ansatz at $q=0$ [9].

We did not intend to make the paper completely self-contained. Exposition of the KKR bijection [8] and Lemma 3.2 have been attributed to [10]. Rather, we have employed a casual description to clarify how the algorithmic solution to the initial value problem [10] leads directly to the explicit formula (3.8). We shall exclusively consider the case where the amplitudes of the solitons are all distinct, which greatly simplifies the presentation. The general case can be treated with the same idea.

## 2. Periodic box-ball system and inverse scattering transform

Let us quickly recall the periodic box-ball system without getting much into the crystal base theory. For a comprehensive treatment, see [10]. For a positive integer $l$, let $B_l=\{(x_1,x_2)\in(\mathbb{Z}_{\geq 0})^2 \mid x_1+x_2=l\}$ and set $u_l=(l,0)\in B_l$. The two elements $(1,0)$ and $(0,1)$ in $B_1$ will be denoted by 1 and 2 for short. (Thus $u_1=1$.) In the following, the symbol $\otimes$ meaning the tensor product of crystals can just be

understood as a product of sets. Define the map $R : B_l \otimes B_1 \to B_1 \otimes B_l$ by

$$(x_1, x_2) \otimes 1 \mapsto \begin{cases} 1 \otimes (l, 0) & \text{if } (x_1, x_2) = (l, 0) \\ 2 \otimes (x_1 + 1, x_2 - 1) & \text{otherwise,} \end{cases}$$

$$(x_1, x_2) \otimes 2 \mapsto \begin{cases} 2 \otimes (0, l) & \text{if } (x_1, x_2) = (0, l) \\ 1 \otimes (x_1 - 1, x_2 + 1) & \text{otherwise.} \end{cases}$$

$R$ is a bijection and called the combinatorial $R$. We write the relation $R(u \otimes b) = b' \otimes u'$ simply as $u \otimes b \simeq b' \otimes u'$, and similarly for any consequent relation of the form $a \otimes u \otimes b \otimes c \simeq a \otimes b' \otimes u' \otimes c$.

A state of the periodic box-ball system is an array of 1 and 2, which is regarded as an element $b_1 \otimes \cdots \otimes b_L \in B_1^{\otimes L}$ with $L$ being the system size. Let the number of $2 \in B_1$ appearing in $b_1 \otimes \cdots \otimes b_L$ be $M$. Without loss of generality we assume $L \geq 2M$ (see [10], section 3.3). Let $\mathcal{P}$ be the set of such states. Then the time evolution $T_l : \mathcal{P} \to \mathcal{P}$ is defined by

$$u_l \otimes p \simeq p^* \otimes v_l, \quad v_l \otimes p \simeq T_l(p) \otimes v_l. \tag{2.1}$$

In the first relation, one applies the combinatorial $R$ for $L$ times to carry $u_l$ through $p \in \mathcal{P}$ to the right. This determines $v_l \in B_l$ and $p^* \in \mathcal{P}$ uniquely. ($p^*$ does not play an essential role.) Then the second relation using the so obtained $v_l$ specifies $T_l(p)$, where the appearance of the same $v_l$ in the right hand side is a non-trivial claim ([10], section 2.2). $v_l$ is dependent on $p$ as opposed to $u_l$.

The combinatorial $R$ is the identity map on $B_1 \otimes B_1$, and therefore $T_1$ is just the cyclic shift $T_1(b_1 \otimes \cdots \otimes b_L) = b_L \otimes b_1 \otimes \cdots \otimes b_{L-1}$. The commutativity $T_l T_k = T_k T_l$ holds for any $k, l$ ([10], Theorem 2.2).

**Example 2.1.** The time evolutions $p, T_l(p), \ldots, T_l^9(p)$ of the state $p$ on the top line are listed downward for $l = 2$ and 3. The system size is $L = 14$. We omit the symbol $\otimes$.

| evolution under $T_2$ | evolution under $T_3$ |
|---|---|
| 1 1 2 1 1 1 2 2 2 1 1 1 2 2 | 1 1 2 1 1 1 2 2 2 1 1 1 2 2 |
| 2 2 1 2 1 1 1 1 2 2 2 1 1 1 | 2 2 1 2 1 1 1 1 1 2 2 2 1 1 |
| 1 1 2 1 2 2 1 1 1 1 2 2 2 1 | 1 1 2 1 2 2 2 1 1 1 1 1 2 2 |
| 2 1 1 2 1 1 2 2 1 1 1 1 2 2 | 2 2 1 2 1 1 1 2 2 2 1 1 1 1 |
| 2 2 2 1 2 1 1 1 2 2 1 1 1 1 | 1 1 2 1 2 2 1 1 1 1 2 2 2 1 |
| 1 1 2 2 1 2 2 1 1 1 2 2 1 1 | 2 2 1 2 1 1 2 2 1 1 1 1 1 2 |
| 1 1 1 1 2 1 2 2 2 1 1 1 2 2 | 1 1 2 1 2 2 1 1 2 2 2 1 1 1 |
| 2 2 1 1 1 2 1 1 2 2 2 1 1 1 | 1 1 1 2 1 1 2 2 1 1 1 2 2 2 |
| 1 1 2 2 1 1 2 1 1 1 2 2 2 1 | 2 2 2 1 2 1 1 1 2 2 1 1 1 1 |
| 2 1 1 1 2 2 1 2 1 1 1 1 2 2 | 1 1 1 2 1 2 2 2 1 1 2 2 1 1 |

Regarding 1 as an empty box and 2 as a ball, these patterns exhibit the nonlinear dynamics of balls. There are three solitons (wavepackets) with amplitudes $3, 2$ and 1 traveling to the right.

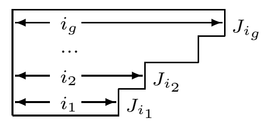
Let us proceed to the direct and inverse scattering transforms. A state $p = b_1 \otimes \cdots \otimes b_L$ is called *highest* if

$$\sharp\{1 \leq i \leq k \mid b_i = 1\} \geq \sharp\{1 \leq i \leq k \mid b_i = 2\} \quad \text{for all } \ 1 \leq k \leq L.$$

The state on the top line in example 2.1 is highest, whereas those on the second lines are not. Let $\mathcal{P}_+$ be the subset of $\mathcal{P}$ consisting of the highest states. Any state

$p \in \mathcal{P}$ can be expressed as $p = T_1^d(p_+)$ using some $d \in \mathbb{Z}$ and a highest state $p_+ \in \mathcal{P}_+$. For instance, the state $T_2(p)$ in example 2.1 is written as $22121111222111 = T_1^2(12111122211122)$. Given a state $p$, such $d$ and $p_+$ are not unique in general. Picking any one of them will be denoted by $p \mapsto (d, p_+)$. Consider the KKR bijection $\phi$ from the highest state $p_+$ to the rigged configuration [8]:

(2.2) $$p_+ \overset{\phi}{\longmapsto}$$




The partition $(i_g, \ldots, i_2, i_1)$ is called the configuration and the integers $0 \le J_i \le p_i$ are called the rigging. The combined data define a rigged configuration. Here $p_i$ is the vacancy number:

$$p_i = L - 2\sum_{j\in\mu} \min(i, j), \tag{2.3}$$

where $\mu = \{i_1 \le i_2 \le \cdots \le i_g\}$. Obviously, $p_{i_1} \ge p_{i_2} \ge \cdots \ge p_{i_g}$ holds, and it is known that $i_1 + \cdots + i_g$ coincides with the number $M$ of $b_k = 2 \in B_1$ contained in $p_+ = b_1 \otimes \cdots \otimes b_L$. Thus we have $p_{i_g} = L - 2M \ge 0$ by the assumption. See appendix A in [10] for an exposition adapted to the present context.

The configuration $\mu$ is actually independent of the non-uniqueness of the choice of $p_+$, and determined solely from $p$. The states are classified according to their configurations:

$$\mathcal{P} = \bigsqcup_{\mu} \mathcal{P}(\mu),$$

where the disjoint union runs over all the partitions of $M = 0, 1, \ldots, [L/2]$. $\mathcal{P}(\mu)$ is the set of states whose configuration is $\mu$. Each subset $\mathcal{P}(\mu)$ is invariant under any time evolution $T_l$, telling us that $\mu$ is a conserved quantity ([10] Corollary 3.5). Physical meaning of $\mu$ is the *soliton content*, namely, the list of the amplitudes of the solitons involved in $p$. In particular $g$ is the number of solitons.

Unless otherwise stated, we shall consider those states whose configuration has the distinct parts as

$$\mu = \{i_1 < i_2 \cdots < i_g\}. \tag{2.4}$$

Define the $g \times g$ symmetric integer matrix $A = (A_{i,j})_{i,j\in\mu}$ and the lattice $\Gamma$ by

$$A_{i,j} = \delta_{i,j} p_i + 2\min(i, j), \qquad \Gamma = A\mathbb{Z}^g \subset \mathbb{Z}^g. \tag{2.5}$$

This matrix has arisen in the Bethe equation at $q = 0$ (4.1) known as the string centre equation [9]. Under the condition $L \ge 2M$, $A$ is positive definite.

Let us proceed to the scattering data, i.e., the action-angle variables. The action variable is the set $\mu$ itself. The set of angle variables with prescribed $\mu$ is given by the quotient:

$$\mathcal{J} = \mathcal{J}(\mu) = \mathbb{Z}^g/\Gamma. \tag{2.6}$$

The one to be assigned with the state $p$ is found by the direct scattering map:

$$\begin{array}{rccccc} \Phi: & \mathcal{P}(\mu) & \longrightarrow & \mathbb{Z}\times\mathcal{P}_+ & \longrightarrow & \mathcal{J}(\mu) \\ & p & \longmapsto & (d, p_+) & \longmapsto & (\mathbf{J} + d\mathbf{h}_1)/\Gamma, \end{array} \tag{2.7}$$

where $\mathbf{h}_1 = (1, \dots, 1) \in \mathbb{Z}^g$ as defined in (2.8). $\mathbf{J} = (J_i)_{i\in\mu} \in \mathbb{Z}^g$ is specified by the KKR bijection as in (2.2), which we write as $\phi(p_+) = (\mu, \mathbf{J})$ or simply $\phi(p_+) = \mathbf{J}$. Then $\mathbf{J}+d\mathbf{h}_1 = (J_i+d)_{i\in\mu}$. $\Phi$ is well-defined [10]. In particular, the non-uniqueness of the decomposition $p \mapsto (d, p_+)$ is cancelled by taking mod $\Gamma$. For $\mathbf{I} \in \mathbb{Z}^g$, we denote its image in $\mathcal{J}$ by the same symbol $\mathbf{I}$.

For $\mathbf{I} \in \mathcal{J}$ we introduce the time evolution through

$$T_l(\mathbf{I}) = \mathbf{I} + \mathbf{h}_l, \quad \mathbf{h}_l = (\min(i,l))_{i\in\mu} \in \mathbb{Z}^g. \tag{2.8}$$

Note that $L\mathbf{h}_1 = A\mathbf{h}_1 \in \Gamma$, therefore $T_1^L(\mathbf{I}) = \mathbf{I} \in \mathcal{J}$.

**Theorem 2.2** ([10], Theorems 3.11, 3.12)**.** *The map $\Phi$ is a bijection and the following commutative diagram is valid:*

$$\begin{tikzcd} \mathcal{P}(\mu) \arrow[r, "\Phi"] \arrow[d, "T_l"'] & \mathcal{J}(\mu) \arrow[d, "T_l"] \\ \mathcal{P}(\mu) \arrow[r, "\Phi"] & \mathcal{J}(\mu) \end{tikzcd} \tag{2.9}$$

*Here $T_l$ on the left and the right are given by (2.1) and (2.8), respectively.*

The composition $\Phi^{-1} \circ T_l \circ \Phi$ yields the algorithmic solution of the initial value problem by the inverse scattering method [6, 1]. The nonlinear dynamics $T_l$ on $\mathcal{P}(\mu)$ becomes the straight motion on $\mathcal{J}(\mu)$ with the velocity $\mathbf{h}_l$. In this sense $\mathcal{J}(\mu)$ is an ultradiscrete analogue of the Jacobi variety. Its cardinality is given by $|\mathcal{J}(\mu)| = \det A = Lp_{i_1}p_{i_2}\cdots p_{i_{g-1}}$ ([10], (4.6),(4.13) and (4.21)). For $l \geq i_g$, one has $\mathbf{h}_l = \mathbf{h}_{i_g}$, hence $T_l(p) = T_{i_g}(p)$ by theorem 2.2.

In the limit $L \to \infty$, the quotient by $\Gamma$ in (2.6) becomes void and the result provides the inverse scattering method for the box-ball system on the infinite lattice. The direct and the inverse scattering maps $\Phi^{\pm 1}$ reduce to the KKR bijection $\phi^{\pm 1}$ itself.

**Example 2.3.** For $p = 22121111222111$, let us derive

$$T_2^{1000}(p) = 11112221112212, \qquad T_3^{1000}(p) = 12211122111122 \tag{2.10}$$

based on the inverse scattering scheme (2.9). (This $p$ is $T_2(p)$ in example 2.1.) We have $p = T_1^2(p_+)$ with the highest state $p_+ = 12111122211122$. The image of the KKR bijection of $\phi(p_+)$ and the direct scattering transform $\Phi(p)$ are given by

$$\phi(p_+) = \begin{array}{l} \square\square\square\ 1 \\ \square\square\ 4 \\ \square\ 0 \end{array} \qquad\qquad \Phi(p) = \begin{array}{l} \square\square\square\ 3 \\ \square\square\ 6 \\ \square\ 2 \end{array}$$

Thus $\mu = \{1, 2, 3\}, (p_1, p_2, p_3) = (8, 4, 2)$ and the matrix $A$ (2.5) reads

$$A = \begin{pmatrix} p_1 + 2 & 2 & 2 \\ 2 & p_2 + 4 & 4 \\ 2 & 4 & p_3 + 6 \end{pmatrix} = \begin{pmatrix} 10 & 2 & 2 \\ 2 & 8 & 4 \\ 2 & 4 & 8 \end{pmatrix}.$$

According to (2.9) and (2.8), the scattering data for the states $T_{2,3}^{1000}(p)$ are given by

$$T_2^{1000}(\Phi(p)) = \begin{array}{l} \square\square\square\ 2003 \\ \square\square\ 2006 \\ \square\ 1002 \end{array} \qquad T_3^{1000}(\Phi(p)) = \begin{array}{l} \square\square\square\ 3003 \\ \square\square\ 2006 \\ \square\ 1002 \end{array}$$

The angle variables appearing here are written as

$$\begin{pmatrix}1002\\2006\\2003\end{pmatrix} = \begin{pmatrix}8\\4\\1\end{pmatrix} + 0\mathbf{h}_1 + A\begin{pmatrix}35\\161\\161\end{pmatrix}, \qquad \begin{pmatrix}1002\\2006\\3003\end{pmatrix} = \begin{pmatrix}6\\0\\1\end{pmatrix} + 4\mathbf{h}_1 + A\begin{pmatrix}17\\81\\330\end{pmatrix}.$$

The last terms involving $A$ can be dropped by $\bmod \Gamma$, whereas the first terms in the right hand sides give rise to the rigged configurations and the corresponding highest states:

$$11112221112212 \xleftarrow{\phi^{-1}} \begin{array}{l} \square\square\square\ 1 \\ \square\square\ 4 \\ \square\ 8 \end{array} \qquad 11221111221221 \xleftarrow{\phi^{-1}} \begin{array}{l} \square\square\square\ 1 \\ \square\square\ 0 \\ \square\ 6 \end{array}$$

In view of $+0\mathbf{h}_1$ and $+4\mathbf{h}_1$, $T_2^{1000}(p)$ and $T_3^{1000}(p)$ are obtained by taking the cyclic shifts $T_1^0$ and $T_1^4$ of these states respectively, in agreement with (2.10).

## 3. The explicit formula for the initial value problem

First we present a piecewise linear formula for the KKR bijection. Let $(\mu, \mathbf{J})$ be a rigged configuration for a highest state in $B_1^{\otimes L}$. To be concrete, we set

$$\phi^{-1}((\mu, \mathbf{J})) = (1 - y(1), y(1)) \otimes \cdots \otimes (1 - y(L), y(L)) \in \mathcal{P}_+,$$

where $y(k) \in \{0, 1\}$ is the 'number of balls' in the $k$ th box from the left. We parametrize the configuration $\mu = \{i_1, \ldots, i_g\}$ and the rigging $\mathbf{J} = (J_{i_1}, \ldots, J_{i_g})$ as in (2.2). The following proposition 3.1 and lemma 3.2 hold for the configurations such that $i_1 \le \cdots \le i_g$.

**Proposition 3.1.** *The image of the KKR bijection is given by*

$$y(k) = \tau_0(k) - \tau_0(k-1) - \tau_1(k) + \tau_1(k-1), \tag{3.1}$$

$$\tau_r(k) = -\min_{\mathbf{n} \in \{0,1\}^g} \Big\{\sum_{i \in \mu} (J_i + ri - k) n_i + \sum_{i,j \in \mu} \min(i,j) n_i n_j\Big\} \quad (r = 0, 1), \tag{3.2}$$

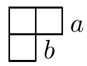
*where* $\mathbf{n} = (n_{i_1}, \ldots, n_{i_g})$.

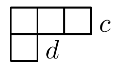
The proof will be given elsewhere for a more general case. $\tau_r(k) \in \mathbb{Z}_{\ge 0}$ is the ultradiscrete tau function mentioned in section 1. We remark that there is *no* dependence on $L$ in (3.2) except in the upper bound $p_i$ (2.3) of the rigging $J_i \le p_i$. For $k < 1$ or $k > L$, (3.1) gives $y(k) = 0$. As it turns out, after theorem 3.3, proposition 3.1 essentially provides the solution of the initial value problem of the box-ball system on the infinite lattice $k \in \mathbb{Z}$.

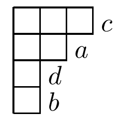
**Lemma 3.2** ([10], Lemma C.1)**.** *Let $q \in B_1^{\otimes K}$ and $r \in B_1^{\otimes L}$ be the highest states associated with the rigged configurations $\phi(q) = (\lambda, \mathbf{I})$ and $\phi(r) = (\mu, \mathbf{J})$. Then the rigged configuration of the highest state $q \otimes r \in B_1^{\otimes K+L}$ is $\phi(q \otimes r) = (\lambda \cup \mu, \mathbf{I} \cup \mathbf{J}')$, where $\mathbf{J}' = (J'_j)_{j \in \mu}$ is given by*

$$J'_j = J_j + p_j, \qquad p_j = K - 2\sum_{k \in \lambda} \min(j, k).$$

The shift $p_j$ here is nothing but the vacancy number in the rigged configuration $\phi(q)$. The notation $(\lambda \cup \mu, \mathbf{I} \cup \mathbf{J}')$ means the union regarding $(\lambda, \mathbf{I})$ and $(\mu, \mathbf{J}')$ as multi-sets of parts (rows in Young diagrams) assigned with rigging. For example,

$(\lambda, \mathbf{I}) =$ 

$(\mu, \mathbf{J}') =$ 

$(\lambda \cup \mu, \mathbf{I} \cup \mathbf{J}') =$ 


where, as usual, the ordering of the rigging $d$ and $b$ within a block of equal length rows does not matter. In what follows, we employ the convention of always arranging the rigging to weakly increase upward within such blocks.

Given a state $p \in \mathcal{P}$, take a highest state $p_+ \in \mathcal{P}_+$ and $0 \le d < L$ such that $p = T_1^d(p_+)$. Let $\phi(p_+) = (\mu, \mathbf{J})$ be the rigged configuration for $p_+$, which we parametrize as $\mu = \{i_1, \ldots, i_g\}$ and $\mathbf{J} = (J_{i_1}, \ldots, J_{i_g})$. Here we assume $i_1 < \cdots < i_g$ in accordance with the assumption (2.4). We form a large highest state $p_+^{\otimes N} = p_+ \otimes \cdots \otimes p_+ \in B_1^{\otimes NL}$. By lemma 3.2, its rigged configuration $(\mu^N, \mathbf{J}^N) := \phi(p_+^{\otimes N})$ is given by

$$
\begin{aligned}
&\mu^N = \{i_{1,1}, \ldots, i_{1,N}, i_{2,1}, \ldots, i_{2,N}, \ldots, i_{g,1}, \ldots, i_{g,N}\},\\
&\mathbf{J}^N = (J_{i_1,1}, \ldots, J_{i_1,N}, J_{i_2,1}, \ldots, J_{i_2,N}, \ldots, J_{i_g,1}, \ldots, J_{i_g,N}),\\
&i_{s,\alpha} = i_s, \qquad J_{i_s,\alpha} = J_{i_s} + (\alpha - 1)p_{i_s} \quad (1 \le \alpha \le N),
\end{aligned}
$$

where $p_i = L - 2\sum_{j \in \mu} \min(i,j)$ is the vacancy number for $p_+$. We apply proposition 3.1 to $(\mu^N, \mathbf{J}^N)$. From (3.2) the corresponding ultradiscrete tau function $\tau_r(k)$ reads

(3.3)
$$
-\min_{\mathbf{n} \in \{0,1\}^{Ng}} \left\{ \sum_{i \in \mu} \sum_{1 \le \alpha \le N} (J_{i,\alpha} + ri - k) n_{i,\alpha} + \sum_{i,j \in \mu} \sum_{1 \le \alpha, \beta \le N} \min(i,j) n_{i,\alpha} n_{j,\beta} \right\},
$$

where $\mathbf{n} = (n_{i_1,1}, \ldots, n_{i_1,N}, \ldots, n_{i_g,1}, \ldots, n_{i_g,N})$. Since $J_{i,1} \le J_{i,2} \le \cdots \le J_{i,N}$ for each $i \in \mu$, the minimum here can be restricted to those $\mathbf{n}$ having the form

$$
n_{i,1} = n_{i,2} = \cdots = n_{i,m_i} = 1, \quad n_{i,m_i+1} = n_{i,m_i+2} = \cdots = n_{i,N} = 0
$$

for some $0 \le m_i \le N$. Then the sums over $\alpha$ and $\beta$ in (3.3) can be taken, leading to

$$
\begin{aligned}
&\sum_{i \in \mu} \Big( m_i J_i + \frac{m_i(m_i - 1)}{2} p_i + m_i r i - m_i k \Big) + \sum_{i,j \in \mu} \min(i,j) m_i m_j \\
&= {}^t\mathbf{m}\big(\mathbf{J} - \frac{\mathbf{p}}{2} + r\mathbf{h}_\infty - k\mathbf{h}_1\big) + \frac{1}{2}{}^t\mathbf{m} A \mathbf{m},
\end{aligned}
\tag{3.4}
$$

where $A = (A_{i,j})$ is defined in (2.5). We have set $\mathbf{m} = (m_i)_{i \in \mu}$, $\mathbf{p} = (p_i)_{i \in \mu}$ and used the vector notation $\mathbf{J}, \mathbf{h}_1, \mathbf{h}_\infty$ around (2.7) and (2.8). For instance $\mathbf{h}_\infty = \mathbf{h}_{i_g}$ and (2.3) is rephrased as

$$
\mathbf{p} = L\mathbf{h}_1 - 2\sum_{j \in \mu} \mathbf{h}_j. \tag{3.5}
$$

By taking $N$ to be even and shifting $\mathbf{m}$ to $\mathbf{m} + \frac{N}{2}\mathbf{h}_1$, (3.4) is rewritten as ${}^t\mathbf{m}(\mathbf{J} - \frac{\mathbf{p}}{2} + r\mathbf{h}_\infty - (k - \frac{NL}{2})\mathbf{h}_1) + \frac{1}{2}{}^t\mathbf{m}A\mathbf{m} + X$, where $X = \frac{N}{2}{}^t\mathbf{h}_1(\mathbf{J} - \frac{\mathbf{p}}{2} + r\mathbf{h}_\infty - (k - \frac{NL}{4})\mathbf{h}_1)$.

This $X$ can be put outside min, after which its dependence on $r, k$ is cancelled in the difference (3.1). Therefore we find that $p_+^{\otimes N} = (1-y(1), y(1)) \otimes \cdots \otimes (1-y(NL), y(NL))$ is given by (3.1) with $\tau_r(k)$ replaced by

$$\tau_r(k) = -\min_{\mathbf{m}}\{{}^t\mathbf{m}(\mathbf{J} - \frac{\mathbf{p}}{2} + r\mathbf{h}_\infty - (k - \frac{NL}{2})\mathbf{h}_1) + \frac{1}{2}{}^t\mathbf{m}A\mathbf{m}\}, \tag{3.6}$$

where min is taken over those $\mathbf{m} = (m_i)_{i\in\mu} \in \mathbb{Z}^g$ such that $-N/2 \leq m_i \leq N/2$.

From the relation $p = T_1^d(p_+)$, the state $p = (1-x(1), x(1)) \otimes \cdots \otimes (1-x(L), x(L))$ is obtained from $p_+^{\otimes N}$ by picking up the length $L$ segment corresponding to $y(wL-d+1), \ldots, y((w+1)L-d)$ for any $1 \leq w \leq N-1$. Thus in (3.6) we replace $k$ by $k + wL - d$ with the choice $w = \frac{N}{2}$ to get $\tau_r(k) = -\min_{\mathbf{m}}\{c_L(\mathbf{m})\}$ with

$$c_L(\mathbf{m}) = {}^t\mathbf{m}\big(\mathbf{I} - \frac{\mathbf{p}}{2} - k\mathbf{h}_1 + r\mathbf{h}_\infty\big) + \frac{1}{2}{}^t\mathbf{m}A\mathbf{m}. \tag{3.7}$$

Here we have let $\mathbf{I} = \mathbf{J} + d\mathbf{h}_1$ denote the angle variable $\Phi(p)$ for $p$. See (2.7). The resulting formula for $x(k)$ gives the state $p$ corresponding to its action-angle variable $(\mu, \mathbf{I})$ *as long as* $0 \leq d \leq L-1$, $1 \leq k \leq L$ and $0 \leq J_i \leq p_i$ since we have started from the rigged configuration. These constraints are removed by taking the limit $N \to \infty$, where the minimum extends over $\mathbf{m} \in \mathbb{Z}^g$; therefore one has

$$\tau_r(k) = \Theta\big(\mathbf{I} - \frac{\mathbf{p}}{2} - k\mathbf{h}_1 + r\mathbf{h}_\infty\big).$$

By virtue of the quasi-periodicity of the ultradiscrete Riemann theta function (1.2), the difference

$$\begin{aligned} x(k) = {} & \Theta\big(\mathbf{I} - \frac{\mathbf{p}}{2} - k\mathbf{h}_1\big) - \Theta\big(\mathbf{I} - \frac{\mathbf{p}}{2} - (k-1)\mathbf{h}_1\big) \\ & - \Theta\big(\mathbf{I} - \frac{\mathbf{p}}{2} - k\mathbf{h}_1 + \mathbf{h}_\infty\big) + \Theta\big(\mathbf{I} - \frac{\mathbf{p}}{2} - (k-1)\mathbf{h}_1 + \mathbf{h}_\infty\big) \end{aligned} \tag{3.8}$$

gains the invariance under $k \to k + L$ and $\mathbf{I} \to \mathbf{I} + \mathbf{v}$ for any $\mathbf{v} \in \Gamma = A\mathbb{Z}^g$. (Note that $L\mathbf{h}_1 = A\mathbf{h}_1 \in \Gamma$.) Namely, (3.8) makes sense for $k \in \mathbb{Z}_L$ and $\mathbf{I} \in \mathcal{J} = \mathbb{Z}^g/\Gamma$.

To summarize, we have proved

**Theorem 3.3.** *For any state $p \in \mathcal{P}$ of the periodic box-ball system, let $(\mu, \mathbf{I}) = \Phi(p)$ be the action-angle variable. Fix $\mathbf{p} = (p_i)_{i\in\mu}$ by (3.5) and the matrix $A$ by (2.5). Then the state $p$ is expressed as $p = (1-x(1), x(1)) \otimes \cdots \otimes (1-x(L), x(L))$ with $x(k) \in \{0, 1\}$ given by (3.8). Due to theorem 2.2, this solves the initial value problem in that any time evolution $T_{l_1}^{\gamma_1} \cdots T_{l_t}^{\gamma_t}(p)$ is obtained by replacing $\mathbf{I}$ in (3.8) with $\mathbf{I} + \gamma_1\mathbf{h}_{l_1} + \cdots + \gamma_t\mathbf{h}_{l_t}$ ($\gamma_i \in \mathbb{Z}$).*

The quadratic form (3.7) is decomposed as $c_L(\mathbf{m}) = L\sum_{i=1}^g m_i(m_i-1)/2 + c(\mathbf{m})$, where $c(\mathbf{m})$ is independent of the system size $L$. In the limit $L \to \infty$, the minimum is restricted to $\mathbf{m} \in \{0,1\}^g$ and $\Theta$ degenerates into the ultradiscrete tau function as in the scheme (1.3). If $\mathbf{I}$ is chosen to be a rigged configuration, the formula (3.8) under such a reduction still describes the image of the KKR bijection although the function $-\min_{\mathbf{m}\in\{0,1\}^g}\{c(\mathbf{m})\}$ takes slightly different form from (3.2). The result provides the solution of the initial value problem of the box-ball system on the infinite lattice.

In Figure 1, we plot the following function on the $(k, t)$ (space-time) plane:

$$u(k,t) = \frac{\vartheta\big(T_\infty^t(\mathbf{I}) - \frac{\mathbf{p}}{2} - k\mathbf{h}_1\big)\vartheta\big(T_\infty^t(\mathbf{I}) - \frac{\mathbf{p}}{2} - (k-1)\mathbf{h}_1 + \mathbf{h}_\infty\big)}{\vartheta\big(T_\infty^t(\mathbf{I}) - \frac{\mathbf{p}}{2} - (k-1)\mathbf{h}_1\big)\vartheta\big(T_\infty^t(\mathbf{I}) - \frac{\mathbf{p}}{2} - k\mathbf{h}_1 + \mathbf{h}_\infty\big)}, \tag{3.9}$$

where $T^t_\infty(\mathbf{I}) = \mathbf{I} + t\mathbf{h}_\infty$ by (2.8) and $\vartheta(\mathbf{z}) = \sum_{\mathbf{n}\in\mathbb{Z}^g} \exp\Big(-({}^t\mathbf{n}A\mathbf{n}/2 + {}^t\mathbf{n}\mathbf{z})/\epsilon\Big)$ is the Riemann theta function. In view of the scheme (1.3), one has $\lim_{\epsilon\to+0}\epsilon\log u(k,0) = x(k)$. Thus $u(k,t)$ gives a softening of the envelop of ultradiscrete solitons in the periodic box-ball system at $\epsilon = 0$ under the time evolution $T_\infty$. The selected parameters are

$$L = 170,\ \mu = \{2,6\}, \mathbf{I} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},\ \mathbf{p} = \begin{pmatrix} p_2 \\ p_6 \end{pmatrix} = \begin{pmatrix} 162 \\ 154 \end{pmatrix},\ A = \begin{pmatrix} 166 & 4 \\ 4 & 166 \end{pmatrix},\ \epsilon = 7.$$

For the periodic box-ball system described by (3.8), this data corresponds to $p = 1122111111222222 \otimes 1^{\otimes 154}$, which is a two soliton state with amplitudes 2 and 6. At $t = 70$, it becomes $T^{70}_\infty(p) = 1^{\otimes 94} \otimes 222222 \otimes 1^{\otimes 38} \otimes 22 \otimes 1^{\otimes 30}$.

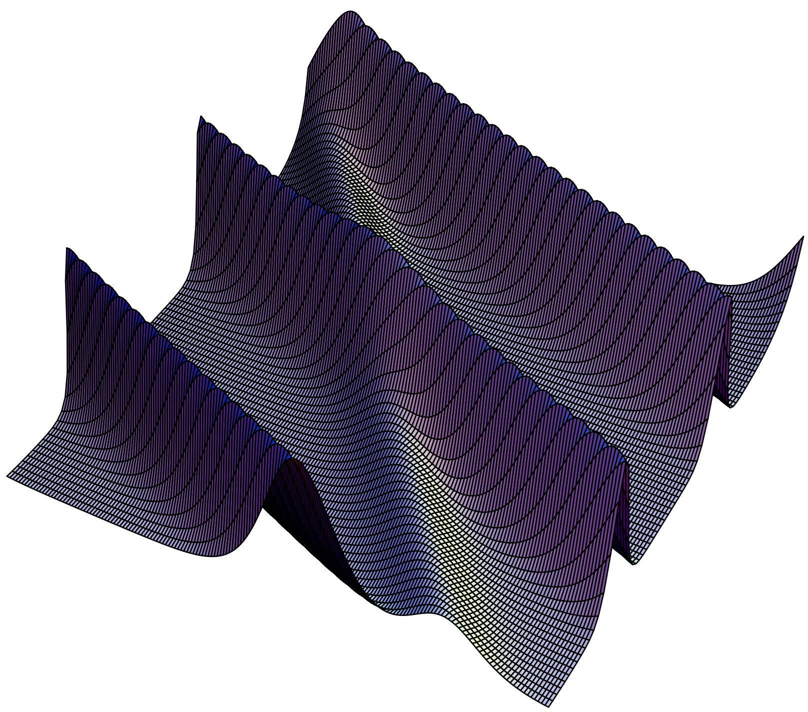


FIGURE 1. The envelope of the function $u(k,t)$ (3.9) for $1 \le k \le 170$, $0 \le t \le 70$. The top and right corners correspond to $(k,t) = (0,0), (170,0)$, respectively. It is periodic in the $k$-direction.

## 4. Discussion

Theorem 3.3 enables one to construct the joint eigenvectors of $T_1, T_2, \ldots$ in $(\mathbb{C}^2)^{\otimes L}$. The result may be regarded as an explicit formula for $q = 0$ Bethe vectors in terms of the ultradiscrete Riemann theta function. We continue assuming that $\mu$ consists of distinct parts as in (2.4).

The Bethe equation for the periodic XXZ chain on $(\mathbb{C}^2)^{\otimes L}$ associated with $U_q(\widehat{sl}_2)$ becomes linear at $q = 0$ under the string hypothesis. The result is known

as the string centre equation [9]:

$$A\mathbf{u} \equiv -\frac{\mathbf{p}}{2} \mod \mathbb{Z}^g, \tag{4.1}$$

where $\mathbf{u} = (u_{i_1}, \ldots, u_{i_g})$ with $u_i$ being the centre of the length $i$ string. We call $\mathbf{u}$ the Bethe root. In this normalization, the Bethe wave function is a rational function of $\exp(2\pi\sqrt{-1}u_i)$; hence $\mathbf{u}$ lives in $(\mathbb{R}/\mathbb{Z})^g$. Thus there is one to one correspondence between the Bethe root $\mathbf{u}$ and the angle variable $\mathbf{J} \in \mathcal{J} = \mathbb{Z}^g/A\mathbb{Z}^g$ via the relation [10]

$$A\mathbf{u} = \mathbf{J} - \frac{\mathbf{p}}{2}.$$

The time evolution $T_l$ of $\mathbf{J}$ (2.8) induces that of the Bethe roots, which is again a straight motion $T_l(\mathbf{u}) = \mathbf{u} + A^{-1}\mathbf{h}_l$ in $(\mathbb{R}/\mathbb{Z})^g$.

At first sight, this appears contradictory, because $T_1, T_2, \ldots$ are fusion transfer matrices at $q = 0$, which should leave the $q = 0$ Bethe vectors invariant up to an overall scalar as well as the relevant Bethe roots. The answer to this puzzle is that the state $p \in B_1^{\otimes L}$ that we are associating to $\mathbf{u}$ or $\mathbf{J}$ by $\Phi(p) = (\mu, \mathbf{J})$ is a *monomial* in $(\mathbb{C}^2)^{\otimes L}$, which is *not* a Bethe vector at $q = 0$ in general.

It is easy to remedy this. In fact, for each Bethe root $\mathbf{u}$ or equivalently $\mathbf{J} = A\mathbf{u} + \frac{\mathbf{p}}{2} \in \mathcal{J}$, one can construct a vector $|\mathbf{J}\rangle \in (\mathbb{C}^2)^{\otimes L}$ that possesses every aspect as a $q = 0$ Bethe vector as follows:

$$\begin{aligned} |\mathbf{J}\rangle &= \sum_{\mathbf{I}\in\mathcal{J}} c_{\mathbf{I},\mathbf{J}}\, p(\mathbf{I}), \\ c_{\mathbf{I},\mathbf{J}} &= \exp\left(-2\pi\sqrt{-1}\, {}^t\mathbf{I}\Big(A^{-1}(\mathbf{J} - \frac{\mathbf{p}}{2}) + \frac{\mathbf{h}_1}{2}\Big)\right), \\ p(\mathbf{I}) &= \begin{pmatrix} 1 - x(1) \\ x(1) \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 1 - x(L) \\ x(L) \end{pmatrix} \in \mathcal{P}(\mu) \subseteq (\mathbb{C}^2)^{\otimes L}, \end{aligned} \tag{4.2}$$

where $x(k) \in \{0, 1\}$ is specified by (3.8). We embed $B_1^{\otimes L}$ into $(\mathbb{C}^2)^{\otimes L}$ naturally and extend $T_l$ to the latter by $\mathbb{C}$-linearity. The vector $p(\mathbf{I})$ here is nothing but the state of the periodic box-ball system appearing in theorem 3.3. It follows that $T_l(p(\mathbf{I})) = p(\mathbf{I} + \mathbf{h}_l)$. Thus from $\mathcal{J} + \mathbf{h}_l = \mathcal{J}$, it is elementary to check

$$\begin{aligned} T_l|\mathbf{J}\rangle &= \Lambda_l(\mathbf{J})|\mathbf{J}\rangle, \\ \Lambda_l(\mathbf{J}) &= c_{-\mathbf{h}_l,\mathbf{J}} = \exp\left(2\pi\sqrt{-1}\, {}^t\mathbf{h}_l\Big(\mathbf{u} + \frac{\mathbf{h}_1}{2}\Big)\right). \end{aligned}$$

The quantity $\Lambda_l(\mathbf{J})$ here exactly coincides with the $q = 0$ Bethe eigenvalue given in equation (4.28) of [10]. Note further that the transition relation (4.2) is inverted as

$$p(\mathbf{I}) = \frac{1}{|\mathcal{J}|} \sum_{\mathbf{J}\in\mathcal{J}} \bar{c}_{\mathbf{I},\mathbf{J}} |\mathbf{J}\rangle,$$

where $\bar{c}_{\mathbf{I},\mathbf{J}}$ denotes the complex conjugate of $c_{\mathbf{I},\mathbf{J}}$. It follows that the space of the $q = 0$ Bethe vectors $|\mathbf{J}\rangle$ coincides with the space of periodic box-ball states $p$ for each prescribed soliton content $\mu$, namely,

$$\bigoplus_{\mathbf{J}\in\mathcal{J}(\mu)} \mathbb{C}|\mathbf{J}\rangle = \bigoplus_{p\in\mathcal{P}(\mu)} \mathbb{C}\, p.$$

Thus we conclude that the approach here bypasses the formidable task of computing the $q \to 0$ limit of the Bethe vectors in general, but leads to the joint

eigenvectors $|\mathbf{J}\rangle$ of $\{T_l\}$. They form a basis of the space having the prescribed soliton content and possess the spectrum $\Lambda_l(\mathbf{J})$ anticipated from the Bethe ansatz at $q = 0$. Moreover $|\mathbf{J}\rangle$ is parametrized explicitly in terms of the ultradiscrete Riemann theta function.

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